Survivability Analysis under Non-Uniform Stochastically Dependent Node Damages

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***Abstract*— In this paper, we analyze survivability of a wireless sensor network under a non-uniform damage model. The analysis assumes statistically dependent damages or node failures. Specifically, the damage process is represented by a non homogeneous Poisson-Boolean model. Network traffic surviving the damage process is analyzed using survivability measures like expected number of surviving node pairs. Numerical results illustrate the analysis.**

I. Introduction

Consider a wireless sensor network of N nodes. We assume that locations of the nodes are known. The sensor nodes perform measurements over a sensor field. These measurements are then communicated among the sensor nodes for various operations, e.g., distributed computation. Let $F_{i,j}$ be the expected traffic (measured using suitable units) between nodes i and j. A damage field destroys a subset of these nodes. We will assume that the traffic that survives the damage process is that between the surviving node-pairs. Our interest in this paper is in the analysis of this surviving traffic. A simple model for survivability analysis in such scenarios is to assume statistically independent damages or failures of nodes. It is easy to see that the assumption of node damages being statistically independent can be quite unrealistic, especially when the damage process affects large geographical areas e.g., power failures, earthquakes, salvo of weapons, nuclear explosions. This dependency is now well known, e.g., [1]–[3]. In [1], a deterministic analysis of the network survivability with failure dependencies is presented. The notion of a dependence graph to model the effects of the damage causing events is introduced. Two non-adjacent nodes in the statistical dependence graph imply that there is no event that can simultaneously destroy both the nodes. The connectivity of the graph, defined as the minimum number of events required to disconnect the network, is the survivability measure. In [2], in addition to the statistical dependence damages are assumed to be multimodal, i.e., the damaged node can be in one of the degraded states, i.e., intermediate states between the fully operative and fully damaged state. Generalized performance measures are defined and approximations to compute them are provided by considering the most probable states.

Many a time, the damage process can be modeled as a coverage process. For example, the damage centers can form a Poisson process with a damage area around the center. A node in the network is damaged if it is '1-covered' (i.e., node lies in at least one damage area) by the damage process. A coverage process is defined as follows. Let $P = \{\zeta_1, \zeta_2, \ldots\}$ be a countable set of points in a d-dimensional Euclidean space and $\{C_1, C_2, \ldots\}$ a countable collection of non empty d-dimensional sets. Then, $\mathcal{C} := \{\zeta_i + C_i : i = 1, 2, \ldots\}$ is a coverage process [4]. Here $\zeta_i + C_i$ denotes the Minkowski sum of $\{\zeta_i\}$ and C_i. If P is a homogeneous Poisson point process and the C_i's are i.i.d. random sets in d dimensions, also independent of P, then $\mathcal{C}$ is called a Poisson-Boolean (PB) model. Then, an obvious model would be to use the homogeneous Poisson-Boolean model for the damage process. It is clear that for this damage model, the probabilities of damage of the nodes are not independent. This is the approach in [3] where the damage process is modeled as a homogeneous PB model and the survival probability of a node pair is obtained. This is then used to obtain the statistics of the surviving traffic. These are then compared with the statistics obtained by assuming that the nodes are damaged independently. The normalized effect of dependence is then obtained as the ratio of the expected total traffic lost under the dependence assumption to the expected total traffic lost under the independence assumption. For different topologies, i.e., star, series and parallel networks the normalized dependence effect is obtained. It is shown that the independence assumption can significantly underestimate the probability that a communication link survives.

In this paper, our interest is to extend the analysis of [3] to a damage process by a non homogeneous PB model. In many situations, a non homogeneous PB model may be a more appropriate description of the damage process. Aiming errors, environmental heterogeneity, the manner of weapon deployment, etc, can result in a non-uniform damage of an area. We will assume that the impact or damage centers form a non homogeneous spatial Poisson process in two dimensions with a spatially varying deterministic density function, $\lambda(x, y)$. The area damaged by each damage center is assumed to be a circle of random i.i.d. radius. This forms a non homogeneous Poisson-Boolean (PB) model and we now analyze the

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survivability of a sensor network when the damage is a non homogeneous PB model. Like in [3], this model is used to obtain the probability that a node pair survives. An important distinction is that now, these probabilities are specific to a node pair and not identical.

The coverage analysis of a non homogeneous Poisson-Boolean (PB) model is presented in [5]. As mentioned above 1-coverage of a point (or node) in two dimensions implies its damage . Thus we can borrow the analysis for point coverage and joint coverage of two points from [5] to analyze the survivability. It is important to mention here that while the application of the techniques developed in [3], [5] is rather straightforward, the numerical evaluations of the quantities of interest take a rather nasty turn. Interestingly, much of the numerical evaluations that we need, have been addressed quite a while ago in the study of damages by ballistic systems. An example of such an analysis is as follows. A salvo of weapons with point of impact modeled by a two-dimensional Gaussian probability density function with a specified mean (the point being aimed) and a variance (aiming error) and a cookie-cutter damage function (the weapon destroying everything inside the disk of crater-radius from its impact-point and leaving everything outside the disk untouched) is used to analyze the probability of damage of a single point target. Alternative models for the distribution of the point of impact (usually circularly symmetric functions) and for the damage area function have also been considered. See [6]–[8] for a good survey for these models. An important distinction between these models and our analysis is as follows. The ballistics analyses consider a fixed number of weapons and the impact points of the weapons are i.i.d. with the specified distribution. In our models, the number of weapons is random. Another difference is that our interest is in the probability of the destruction of node pairs while ballistics analyses usually considers a single point target. However, the numerical expressions will have a similar flavour and we borrow few results from [7]. Ballistics analyses for multiple point targets and area targets are also available [7], [8].

The rest of the paper is organized as follows. In Section II, the non homogeneous Poisson-Boolean (PB) damage process is described. The joint probabilities of damage and survival of two nodes are obtained. These probabilities are used to obtain the survivability measures in Section III. Numerical illustration of the results developed is provided for the 'Gaussian' damage density function. We conclude the paper with a discussion in Section IV.

II. Preliminaries

We consider a sensor network of N nodes, each node is located in a two-dimensional plane with known location. Let N_i denote Node $i, i = 1, 2, \ldots, N$ and also its location (x_i, y_i) in the x-y plane. We assume that the damage process forms a non homogeneous PB model. Let D_i denote the damage center i and also it's location in $\Re^2$, the x-y plane. Let $D_{i\{i>0\}}$ form a non homogeneous Poisson process with a spatially varying deterministic density function, $\lambda(x, y)$. Recall that the area damaged by D_i is assumed to be a circle of random i.i.d. radius. Let the random radius of the damage circle for the damage center i be denoted by R_i. We assume a cookie-cutter damage function in which the every point in the circle is damaged and every point outside the circle is left untouched. Without loss of generality we assume support for the random radius to be $(0, 1]$. See Fig. 1 for an illustration. A node is assumed to be damaged if it is within at least one damage circle, i.e., 1-coverage of the node. For N_i, define the following indicator variable.

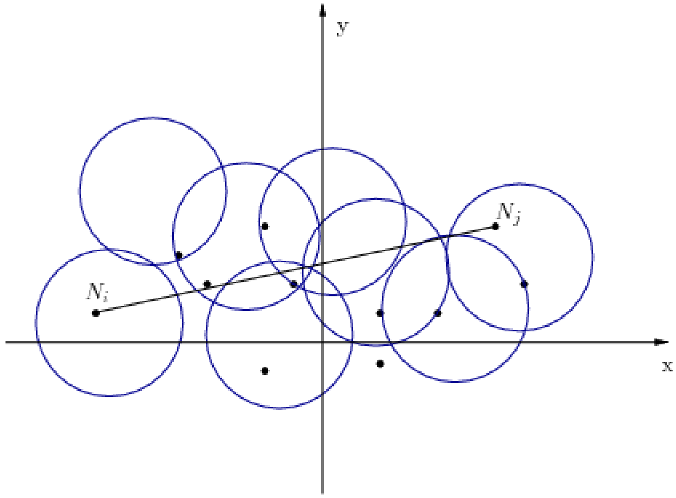

Fig. 1. An illustration of a non-uniform damage process. Dots represent the sensor nodes and circles are the damage circles with their centers being the damage center locations.

$$X_i := \begin{cases} 1, & \text{if } N_i \text{ is damaged,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $p_i = \mathsf{Pr}\left(X_i = 1\right)$. p_i is the probability that N_i is 1-covered. From [5], the number of damage centers that 1-cover N_i is Poisson distributed with mean $m(N_i)$ and

$$p_i = 1 - e^{-m(N_i)} \quad , \tag{1}$$

where $m(N_i)$ is the expected number of damage centers covering N_i.

We now analyze the joint survival and joint damage of the node pair N_i and N_j, located at (x_i, y_i) and (x_j, y_j), respectively. Let

$$\begin{aligned} \Delta_{i,j} &:= \mathsf{Pr}\left(X_i = 1 \text{ and } X_j = 1\right) \quad . \\ \sigma_{i,j} &:= \mathsf{Pr}\left(X_i = 0 \text{ and } X_j = 0\right) \quad . \end{aligned}$$

Let $\tilde{m}(N_i, N_j)$ be the expected number of damage centers (Poisson points) covering N_i and not covering N_j. Further, let $\alpha(N_i, N_j)$ be the expected number of damage centers covering both N_i and N_j. Then $\sigma_{i,j}$ can be obtained from [5] as follows.

$$\begin{aligned} \sigma_{i,j} &:= \mathsf{Pr}\left(N_i \text{ and } N_j \text{ not 1-covered}\right) \\ &:= \mathsf{Pr}\left(N_i \text{ and } N_j \text{ survive}\right) \\ &= e^{-(\tilde{m}(N_i,N_j)+\tilde{m}(N_j,N_i)+\alpha(N_i,N_j))} \end{aligned} \tag{2}$$

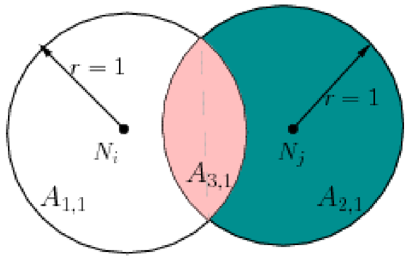

Fig. 2. Joint damage of points N_i and N_j by damage centers of unit damage radius. Damage centers in $A_{1,1}$ damage only N_i, those in $A_{2,1}$ damage only N_j and those in $A_{3,1}$ damage both N_i and N_j.

The results obtained in [5] are general and can be used for any straight line path by a suitable transformation of $\lambda(x, y)$. For example, if our interest is in the straight line segment from (x_i, y_i) to (x_j, y_j), the intensity function used for results will be $\lambda(x_i + x\cos\phi + y\sin\phi, y_i + y\cos\phi - x\sin\phi)$ where $\phi = \frac{y_j - y_i}{x_j - x_i}$. Thus the joint probability for N_i and N_j in two dimensions can be obtained from [5].

We now obtain $\Delta_{i,j}$, the joint probability of damage of the node pair. Fig. 2 illustrates the areas with damage centers covering only N_i, only N_j and both when the damage by each damage center is assumed to be a circle of unit radius. We now define the following events.

- E_1 is the event that there is at least one damage center each in the areas $A_{1,1}$ and $A_{2,1}$. The probability of this event is

$$\mathsf{Pr}\left(E_1\right) = (1 - e^{-\tilde{m}(N_i,N_j)})(1 - e^{-\tilde{m}(N_j,N_i)}) \quad .$$

- E_2 is the event that there is at least one damage center in the area $A_{3,1}$. The probability of E_2 is

$$\mathsf{Pr}\left(E_2\right) = 1 - e^{-\alpha(N_i,N_j)} \quad .$$

The occurrence of either of these two events ensures 1-coverage of both of the nodes. Hence $\Delta_{i,j} = \mathsf{Pr}\left(E_1 \cup E_2\right)$. Since E_1 and E_2 are independent, we have

$$\begin{aligned} \Delta_{i,j} &= (1 - e^{-\tilde{m}(N_i,N_j)})(1 - e^{-\tilde{m}(N_j,N_i)}) \\ &\quad +(1 - e^{-\alpha(N_i,N_j)}) - \\ &\quad (1 - e^{-\tilde{m}(N_i,N_j)})(1 - e^{-\tilde{m}(N_j,N_i)})(1 - e^{-\alpha(N_i,N_j)}) \\ &= 1 - e^{-(\tilde{m}(N_i,N_j)+\tilde{m}(N_j,N_i)+\alpha(N_i,N_j))} \\ &\quad -e^{-m(N_i)} - e^{-m(N_j)} \quad . \end{aligned} \tag{3}$$

Note that the probability that at least one of N_i or N_j is damaged is given by $(1 - \sigma_{i,j})$ whereas the probability that at least one of N_i or N_j survives is $(1 - \Delta_{i,j})$.

III. Survivability Analysis

Let the total number of nodes damaged be denoted by $S = \sum_{i=1}^{N} X_i$. The expectation of S is

$$\begin{aligned} \mathsf{E}\left(S\right) &= \sum_{i=1}^{N} \mathsf{E}\left(X_i\right) \\ &= \sum_{i=1}^{N} p_i \quad . \end{aligned} \tag{4}$$

The expected number of surviving node pairs is

$$\begin{aligned} \mathsf{E}\left(P\right) &= \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathsf{E}\left((1 - X_i)(1 - X_j)\right) \\ &= \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sigma_{i,j} \quad . \end{aligned} \tag{5}$$

Since $1 - X_i$ is also an indicator variable we get (5). Recall that $F_{i,j}$ denotes the expected traffic between nodes i and j. We can then obtain the expected traffic that survives in the damaged network as $\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sigma_{i,j} \times F_{i,j}$.

A. Numerical Study

We now numerically evaluate the survivability measures discussed above when the damage density function is a 'Gaussian' function, i.e., $\lambda(x, y) = \kappa e^{-\theta((x-x^\star)^2+(y-y^\star)^2)}$. $(x^\star, y^\star)$ is the 'center' of the density, $\kappa, \theta > 0$ with κ denoting the damage intensity parameter and θ denoting the decay parameter. Our numerical results assume that the expected number of damage centers, $C = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda(x, y)dx\,dy, = \frac{\kappa\pi}{\theta}$ is kept constant (=100) while varying θ and κ. For the numerical results, we assume that the area damaged by each damage center is a circle of unit radius. We specialize the damage probabilities for individual nodes and for the node pairs obtained in the previous section for the above assumptions.

Recall that $m(N_i)$ is the expected number of damage centers covering N_i. When the damage radii, $R_i = 1$ for all i, from [5], $m(N_i)$ is given by

$$m(N_i) = \iint_{C_{N_i}} \lambda(x, y)\; dx\; dy \quad , \tag{6}$$

where C_{N_i} is the unit circle around N_i. The evaluation of $\sigma_{i,j}$ requires us to integrate the $\lambda(x, y)$ over circles centered at N_i (to obtain $m(N_i)$) and at N_j (to obtain $m(N_j)$). It is difficult to numerically evaluate these integrals. Luckily, there is significant work in this area and the following method is available. To evaluate $m(N_i)$ (and $m(N_j)$), we can use the approach proposed in [7] for circularly symmetric $\lambda(x, y)$ which is based on conversion to polar coordinates. Thus $m(N_i)$ can be obtained as

$$m(N_i) = 2\pi\kappa e^{-\theta R_0^2} \int_0^1 r e^{-\theta r^2} I_0(2\;\theta\; r\; R_0)\; dr \quad ,$$

where, $I_0(\cdot)$ is a modified Bessel function of the first kind of order zero and

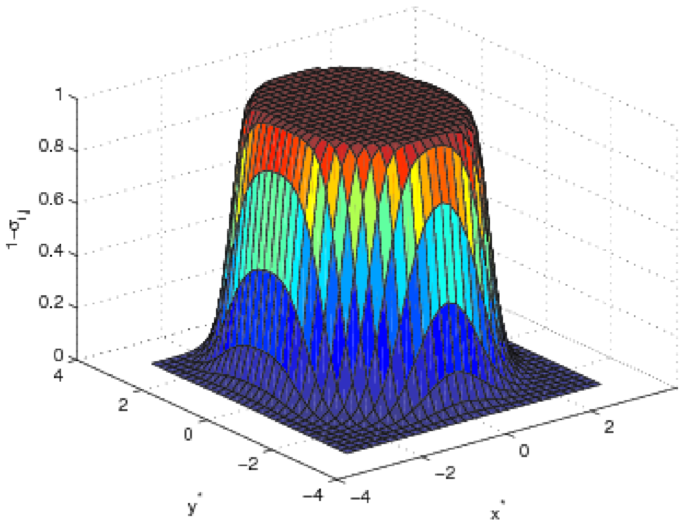

Fig. 3. Probability of damage of at least one point from $(x_i, y_i) = (0, 0)$ and $(x_j, y_j) = (1, 0)$ as a function of location of the center of the damage density, $(x^\star, y^\star)$ when $\theta = 2$.

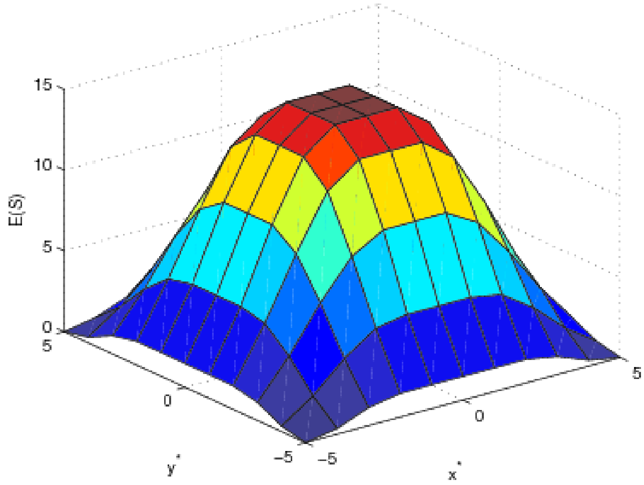

Fig. 4. Expected number of nodes damaged, $\mathsf{E}(S)$ as a function of location of the center of the damage density, $(x^\star, y^\star)$ when $\theta = 2$.

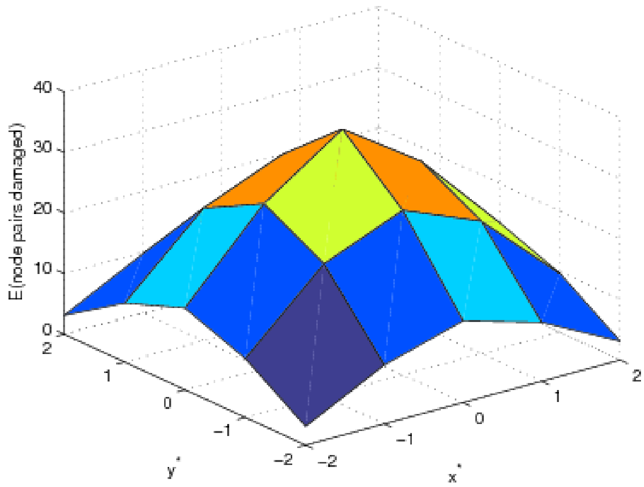

Fig. 5. Expected number of nodes pairs damaged as a function of location of the center of the damage density, $(x^\star, y^\star)$ when $\theta = 2$.

$R_0 = \sqrt{(x^\star - x_i)^2 + (y^\star - y_i)^2}$. We also need to evaluate integration of $\lambda(x, y)$ over the intersection of the two circles to obtain $\alpha(N_i, N_j)$. It is difficult to get a closed form expression for this integral. We evaluate it using polar co-ordinates and geometric methods. From $m(N_i), m(N_j), \alpha(N_i, N_j)$, we can obtain $\mathsf{E}(S), \sigma_{i,j}$ and hence the expected surviving traffic for the given $F_{i,j}$.

It is of interest to study the effects of varying $(x^\star, y^\star)$ and θ on the damage probabilities and hence on the expected number of nodes and node pairs damaged. We first illustrate the effect of varying $(x^\star, y^\star)$ for fixed θ. We plot the probability of damage of at least one of the nodes from the node pair, N_i and N_j, i.e., $(1-\sigma_{i,j})$, as a function of $(x^\star, y^\star)$ as shown in Fig. 3. Here, we assume N_i and N_j are located at $(x_i, y_i) = (0, 0)$ and $(x_j, y_j) = (1, 0)$, respectively and $\theta = 2$.

Now consider a square grid from -3 to 3 units with the step of one unit and the nodes are assumed to be located at every cross-point of this grid. Thus total number of nodes, $N = 49$. We then study the effect of moving the location of the damage center, $(x^\star, y^\star)$ over the cross points of the square grid from -5 to 5 units when $\theta = 2$. Fig. 4 shows how the expected number of nodes damaged, $\mathsf{E}(S)$, varies as a function of $(x^\star, y^\star)$. Clearly, $\mathsf{E}(S)$ decreases as $(x^\star, y^\star)$ moves away from the origin. Fig. 5 shows the expected number of node pairs damaged as a function of $(x^\star, y^\star)$. Here, the nodes are located at the cross-points of the square grid from -1 to 1 units and the location of the damage center is varied over the cross points of the grid from -2 to 2 units. Since our interest is in the expected surviving traffic we consider damage of a node pair as the damage of at least one of the nodes from the node pair.

We now study effect of varying θ on the expected number of node pairs damaged when $(x^\star, y^\star)$ is fixed at $(0, 0)$. We obtain the plots for different values of C (which is the expected number of damage centers) as shown in Fig. 6. Here, the nodes are assumed to be at the cross-points of the square grid from -1 to 1. It can be clearly seen that for small C, the expected number of node pairs damaged is less. Moreover, variation of the expected damaged node pairs as a function of θ is significantly different for different C. It is also important to note that the expected number of node pairs damaged does not vary linearly with C, specifically for smaller values of θ.

IV. Discussion and Future Work

In this paper, we have analyzed the survivability of a sensor network when damage to the network is a non homogeneous Poisson-Boolean (PB) model. Specifically, it is applicable for sensor networks with military applications like battlefield

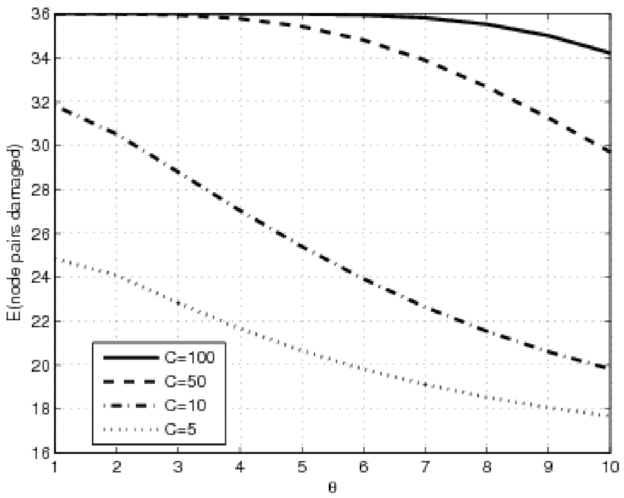

Fig. 6. Expected number of nodes pairs damaged as a function of θ for different values of the expected number of damage centers, C.

surveillance, intruder detection. We have analyzed a stochastic damage model where node failures are statistically dependent and non identical. We have obtained the expected traffic surviving the damaged network from the key result of joint probability of survival of a node pair. We have presented a numerical study on the impact of parameters of the damage density function on the survivability measures.

We have used a damage model where 1-coverage of a node implies the node failure and its survival otherwise. This can be extended to a damage model with multimodal node failures.

Further, the analysis presented in this paper is quite general and can be applied to analyse survivability of other networks like a communication network when the underlying damage process forms a non homogeneous Poisson-Boolean model.

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